MTH 511a - 2020: Lecture 8 - extra

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Ratio-of-Uniforms - the region C

There was some good discussion on the RoU region C. Particularly, there was a discussion on the v axis and whether v can ever be entirely negative or entirely positive. At the time, I stated that if the support of f was entirely negative then v is probably entirely negative. **This is not true!**. Below is a sketch of a proof that $(0,0)$ is always a boundary point of C. This implies that any box around C must be of the form $[a_1, a_2] \times [b_1, b_2]$ such that $a_2 > 0$ and $0 \in [b_1, b_2]$.

The proof below is courtesy Chirag Jindal and another attempt was made by Sanket Agarwal. Well done Chirag and Sanket!

Let $\mathcal{X}$ be the support of f. Recall

$$C = \left\{ (u, v) : 0 \leq u \leq \sqrt{f\left(\frac{v}{u}\right)} \right\} .$$

For $v = 0$,

$$u : 0 \leq u \leq \sqrt{f\left(\frac{v}{u}\right)} = \sqrt{f(0)}$$

- If $0 \in \mathcal{X}$, then $f(0) > 0$, then there exists a $u' \in (0, \sqrt{f(0)})$ such that $(u', 0) \in C$.
- If $0 \notin \mathcal{X}$, then $f(0) = 0$ implying $u = 0$. But since $0/0$ is undefined, this argument is not straightforward.

  Let $(\epsilon, v) \in C$ such that $\epsilon > 0$. Set $k = v/\epsilon$ so that $(\epsilon, k\epsilon) \in C$. Letting $\epsilon \to 0$, we get that $(0, 0)$ is a boundary point of C.

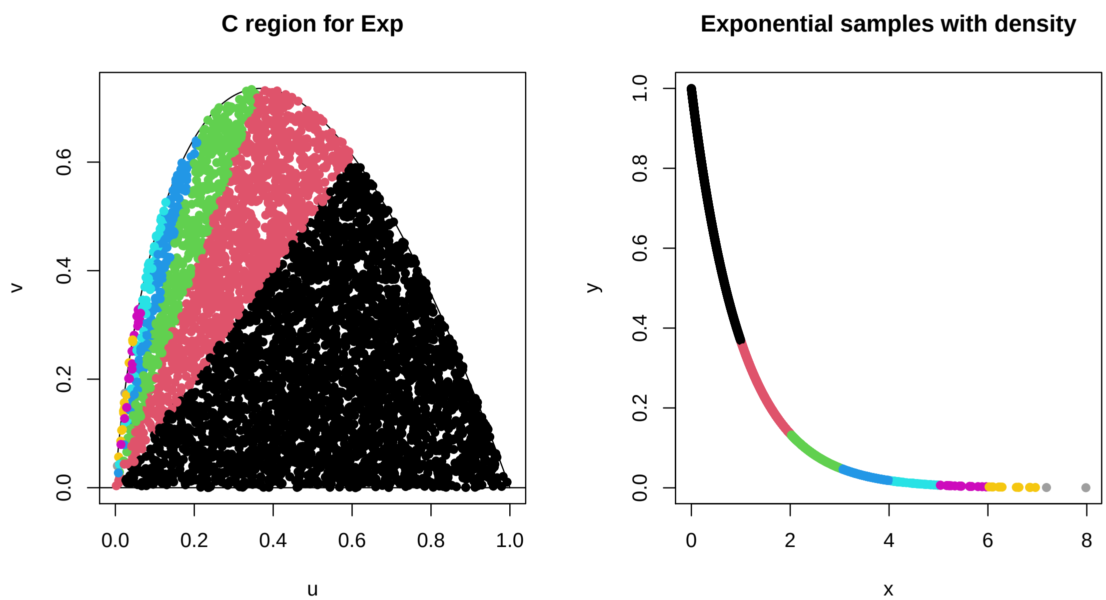
Example 1 (Exponential(1)).

$$f(x) = e^{-x} \quad x \geq 0$$

Here,

$$C = \left\{(u, v) : 0 \leq u \leq e^{-v/2u}\right\}.$$

The region C then is described by $0 \leq u \leq 1$ and $v \leq -2u \log u$. Below is a plot of the region C and uniform samples from C color-coded with how they match with the support of f.



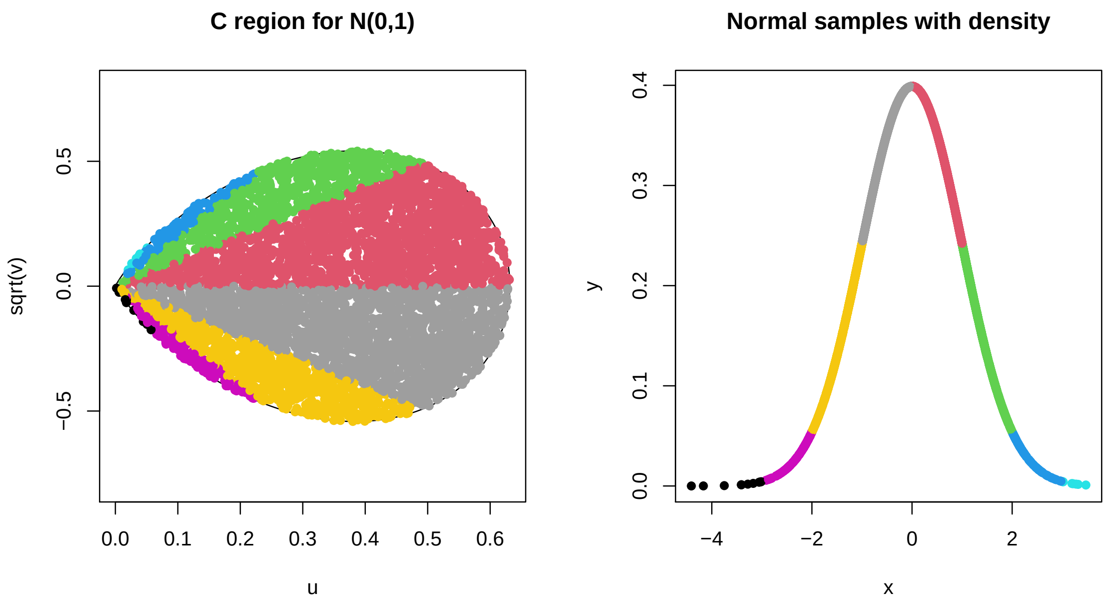
Example 2 (Normal(0,1)). The target density is:

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}.$$

The set C is

$$C = \left\{(u, v) : 0 \leq u \leq \left(\frac{1}{2\pi}\right)^{1/4} e^{-v^2/4u^2}\right\}$$

The region C is defined by $0 \leq u \leq 2\pi^{-1/4}$ and $|v| \leq \sqrt{-4u^2 \log u + \frac{1}{4} \log 2\pi}$. Below is a plot of the region C and uniform samples from C color-coded with how they match with the support of f.


C region for N(0,1)
sqrt(v)
0.5
0.0
−0.5
0.0
0.1
0.2
0.3
0.4
0.5
0.6
u
Normal samples with density
y
0.4
0.3
0.2
0.1
0.0
−4
−2
0
2
x